

# GFD I: Midterm Exam Solutions

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1(a) [2 points] The initial condition is not in geostrophic balance, because it has a velocity field but no surface height changes.

1(b) [2 points] We may drop all terms with  $\partial/\partial y$ , leaving:

$$\text{X-MOM} \quad u_t - fv = -g\eta_x$$

$$\text{Y-MOM} \quad v_t + fu = 0$$

$$\text{MASS} \quad \eta_t + Hu_x = 0$$

1(c) [3 points] Taking the curl of the momentum equations we find:

$$\zeta_t + fu_x = 0$$

And substituting for  $u_x$  using the MASS equation, this becomes:

$$\frac{\partial}{\partial t} \left( \zeta - \frac{f}{H} \eta \right) = 0$$

Then, integrating this in time from  $t = 0$  to some arbitrary later time, we find:

$$\zeta - \frac{f}{H} \eta = \zeta|_{t=0} = -v_0 k \cos(kx) \quad (*)$$

which is the desired expression.

1(d) [3 points] Since the final steady state is geostrophic it will have velocities governed by the steady forms of the momentum equations:

$$\text{X-MOM} \quad v_g = \frac{g}{f} \eta_x$$

$$\text{Y-MOM} \quad u_g = 0$$

Thus the final steady state will be governed by (\*) with the vorticity given by

$\partial v_g / \partial x = (g/f) \eta_{xx}$ , and so the governing equation may be written entirely in terms of

$\eta$ , giving:

$$\eta_{xx} - \frac{1}{a^2}\eta = -\left[\frac{v_0fk}{g}\right]\cos(kx) \quad (**)$$

where  $a \equiv \sqrt{gH}/f$  is the Rossby radius of deformation.

1(e) [5 points] The solution to (\*\*) is given by the sum of solutions to the homogenous equation [(\*\*) with zero on the right hand side] and a particular solution to (\*\*). The boundary conditions for this are just that the solution should be bounded at  $x \rightarrow \pm\infty$ .

Since solutions to the equation  $\eta_{xx} - a^{-2}\eta = 0$  are of the form

$$\eta_{HOMOG} = C_1e^{x/a} + C_2e^{-x/a}$$

The boundary conditions require that  $C_1 = C_2 = 0$ . An obvious form to guess for the particular solution (which is now the complete solution) is:

$$\eta = A\cos(kx)$$

The solution is finished by solving for the unknown constant  $A$  by plugging the solution into (\*\*), which gives (after a little rearrangement):

$$\eta = \frac{v_0}{gk/f} \left[ \frac{(ka)^2}{(ka)^2 + 1} \right] \cos(kx)$$

1(f) [2 points] The final velocity field is found using the geostrophic balance, giving:

$$v = \frac{g}{f}\eta_x = -v_0 \left[ \frac{(ka)^2}{(ka)^2 + 1} \right] \sin(kx)$$

1(g) [5 points] The expression in square brackets is bounded above and below according to:

$$0 \leq \left[ \frac{(ka)^2}{(ka)^2 + 1} \right] \leq 1$$

And so the magnitude of the final vorticity (I mean relative vorticity throughout this discussion) will be less than that of the original vorticity. At the location  $x = 0$  for example, the initial vorticity is negative, and the final vorticity will also be negative, but

smaller in absolute value, meaning that the change in vorticity was positive. This is consistent with the change in surface height at this location, where the surface height raised, so the water column was stretched. The funny thing about this problem is that because there was an initial disturbance in the vorticity field, you can't just look at the final sign of the vorticity to diagnose stretching or squashing, you have to look at the sign of the change in vorticity.

1(h) [2 points] Integrating Y-MOM in time from the initial time to the final time we find:

$$\int_{t_0}^{t_0+\Delta t} [v_t = -fu] dt \Rightarrow -\frac{1}{f} \Delta v = \Delta x = -\frac{1}{f} v_0 \left[ \frac{1}{(ka)^2 + 1} \right] \sin(kx)$$

So, for example, just to the negative side of  $x = 0$ , where  $\Delta v$  was negative, fluid parcels would have moved slightly to the positive-x direction. Likewise, just to the positive side of  $x = 0$ , where  $\Delta v$  was positive, fluid parcels would have moved slightly to the negative-x direction. This is consistent with the fact that the final surface height was raised at  $x = 0$  (by lateral convergence). The scale of this displacement relative to the Rossby radius is given by  $v_0 / \sqrt{gH}$  (times the order-one expression in square brackets above on the right), which is always a small number for large scale atmosphere and ocean flows, therefore parcel displacements will be small relative to  $a$ .

1(i) [3 points] The ratio  $KE_A / PE_A$  of the final flow is given by

$$\left\langle \frac{KE_A}{PE_A} \right\rangle = \frac{\frac{1}{2} \rho H \langle v^2 \rangle}{\frac{1}{2} \rho g \langle \eta^2 \rangle} = \frac{H}{g} \frac{v_0^2}{v_0^2 (gk/f)^{-2}} = (ka)^2$$

where angle brackets denote averaging in  $x$ . This result is true for any SWE flow in geostrophic balance. It shows that as the horizontal length scale of the flow,  $k^{-1}$  in this case, becomes greater than the Rossby radius, the total energy is dominated by potential energy. This result will be particularly important as we consider stratified flows, for which the Rossby radius is much smaller.

1(j) [3 points] The ratio of the total energy of the final state to that of the initial state is given by

$$\left\langle \frac{(KE_A + PE_A)_{\text{FINAL}}}{(KE_A + \cancel{PE_A})_{\text{INITIAL}}} \right\rangle = \frac{\left[ \frac{(ka)^2}{(ka)^2 + 1} \right]^2 \langle KE_{A \text{ INITIAL}} \rangle + \left[ \frac{(ka)^2}{(ka)^2 + 1} \right]^2 \langle KE_{A \text{ INITIAL}} \rangle \frac{1}{(ka)^2}}{\langle KE_{A \text{ INITIAL}} \rangle} = \frac{(ka)^2}{(ka)^2 + 1}$$

Thus when  $ka = 1$ , only half of the original energy remains. When the disturbance scale  $k^{-1}$  is much greater than  $a$ , then  $ka \ll 1$  and the energy remaining approaches zero.

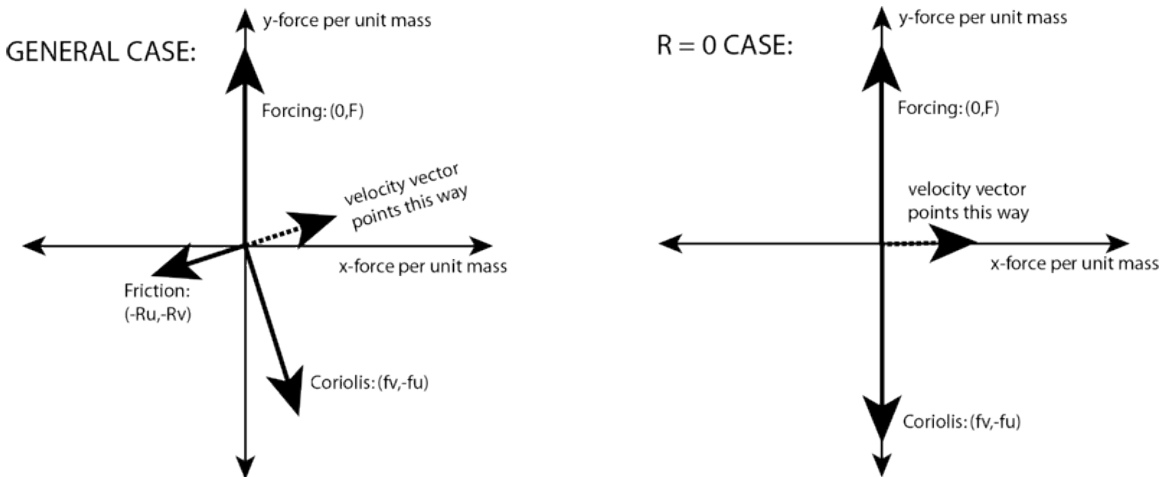
NOTE: If you are interested in exploring the transient motions in this problem, you could download the code “g\_adjust.m” which integrates the equations numerically in time.

2(a) [5 points] In the final steady state the force balance (per unit mass) will just be given by the steady forms of the equations, arranged in the form “acceleration = force/mass.”

$$\text{X-MOM} \quad \cancel{\ddot{x}}_i = fv - Ru$$

$$\text{Y-MOM} \quad \cancel{\ddot{y}}_i = -fu + F - Rv$$

So a sketch of the force balance looks like:



And note that the steady velocities are:

$$u = \frac{F/f}{1+(R/f)^2} \quad \text{and} \quad v = \frac{R}{f}u$$

2(b) [10 points] To solve for the full time-dependent behavior, form a single equation in the quantity  $U = u + iv$ . To do this, just form X\_MOM +  $i$ \*Y-MOM. This is just a mathematical trick to simplify the algebra. The resulting equation is:

$$U_t + (R + if)U = iF$$

The solution is the sum of particular and homogenous solutions, subject to the initial condition  $U(t=0) = 0 + i \times 0$ . The particular solution is easy to guess:

$$U_{\text{PARTICULAR}} = \frac{iF}{R + if} = \frac{Ff + iFR}{R^2 + f^2}$$

The homogenous equation is

$$U_t + (R + if)U = 0$$

For which a suitable guess is  $U_{\text{HOMOGENEOUS}} = (U_1 + iU_2)e^{-\alpha t}$ , where  $\alpha = R + if$ , and  $U_1$  and  $U_2$  are real constants, yet to be determined. Thus the full solution is

$$U = U_{\text{HOMOGENEOUS}} + U_{\text{PARTICULAR}} = (U_1 + iU_2) \left[ \cos(-ft) + i \sin(-ft) \right] e^{-Rt} + \frac{Ff + iFR}{R^2 + f^2}$$

Then you proceed by determining the unknown constants  $U_1$  and  $U_2$  using the initial condition (zero flow at  $t = 0$ ). The velocity components are then just the real and imaginary parts of  $U$ , which may be written (after a bit of algebra) as

$$u = U_0 \left[ \left( -\cos(ft) - \frac{R}{f} \sin(ft) \right) e^{-Rt} + 1 \right]$$

$$v = U_0 \left[ \left( -\frac{R}{f} \cos(ft) + \sin(ft) \right) e^{-Rt} + \frac{R}{f} \right]$$

Where  $U_0 = (F/f) / \left[ 1 + (R/f)^2 \right]$  is a scale for final steady velocity.

2(c) [5 points] To plot the trajectories you have to integrate the velocities over time:

$$x = \int u dt + \text{const.}$$

$$y = \int v dt + \text{const.}$$

In doing this, it is useful to note that

$$\int \cos(ft) e^{-Rt} dt = \text{Re} \left\{ \int e^{(-R+if)t} dt \right\} = \frac{1/f}{1+(R/f)^2} \left[ -\frac{R}{f} \cos(ft) + \sin(ft) \right] e^{-Rt} \equiv G_1$$

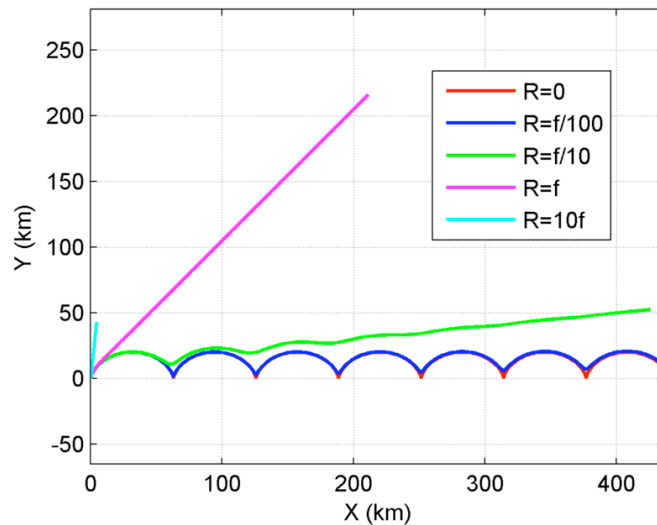
$$\int \sin(ft) e^{-Rt} dt = \text{Im} \left\{ \int e^{(-R+if)t} dt \right\} = \frac{1/f}{1+(R/f)^2} \left[ -\cos(ft) - \frac{R}{f} \sin(ft) \right] e^{-Rt} \equiv G_2$$

Using these to evaluate the integrals of the velocities we find the equations for the particle displacements:

$$x = U_0 \left( -G_1 - \frac{R}{f} G_2 \right) + U_0 t + \text{const.}$$

$$y = U_0 \left( -\frac{R}{f} G_1 + G_2 \right) + \frac{R}{f} U_0 t + \text{const.}$$

And then you just set the constants so that  $x = y = 0$  at  $t = 0$ . You can plot these trajectories using the MATLAB code “midterm\_2.m” which can be downloaded from the class website. Solutions for various values of  $R/f$  look like:



Note that for  $R = 0$  there is a persistent motion to the right, normal to the forcing direction (as we expect for rotating flow) and that superimposed on this are inertial oscillations. The result is a “cycloid.” For greater friction, the inertial oscillations are damped out, and friction allows some steady motion in the direction of forcing.